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MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Phil. Trans. R. Soc. Lond. A 1941 **239**, 217-232
doi: 10.1098/rsta.1941.0002

THE ASYMPTOTIC EXPANSION OF INTEGRAL FUNCTIONS DEFINED BY TAYLOR SERIES (SECOND PAPER)

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(*Communicated by G. H. Hardy, F.R.S.—Received* 27 *July* 1940)

In a former paper I deduced the asymptotic expansion of the integral function

$$f(x)=\sum_{n=0}^{\infty}\frac{\phi(n)\,x^n}{\Gamma(\kappa n+\beta)}\qquad \mathscr{R}((\kappa)>0)$$

for large x from asymptotic properties of the function $\phi(t)$. In particular, $\phi(t)$ had to be regular and its asymptotic behaviour had to satisfy a certain 'condition A' throughout the half-plane $\mathscr{R}(\kappa t)>K$. My results included as special cases most of the known results about the asymptotic expansion of integral functions.

In the present paper the class of functions is widened and the previous theorems completed. I also show that my results are now, in an obvious sense, best possible; that is, if the conditions stated in my theorems are further relaxed, the conclusions are false.

1. In the paper (Wright 1940) to which this is a sequel, I studied the asymptotic expansion of the integral function

$$f(x)=\sum_{n=0}^{\infty}\frac{\phi(n)\,x^n}{\Gamma(\kappa n+\beta)}\qquad (\mathscr{R}(\kappa)>0)$$

for large x when $\phi(t)$ satisfies certain conditions. Among other results, I showed that we could determine the expansion of $f(x)$ wherever this consists of an exponentially large expansion, or of more than one such expansion, provided that $\phi(t)$ is regular and satisfies a certain 'condition A' in the half-plane* $\mathscr{R}(\kappa t)>K$.

My purpose here is to calculate the asymptotic expansion of $f(x)$ for part at least of the x-plane when our hypotheses are less, viz. that $\phi(t)$ is regular and satisfies condition A throughout the 'sector'

$$-\mu_1\leqslant\arg\kappa t\leqslant\mu_2\qquad |t|>K, \tag{1·1}$$

where μ_1, μ_2 are any real numbers subject only to the condition that†

$$-\tfrac{1}{2}\pi<-\mu_1\leqslant\gamma\leqslant\mu_2<\tfrac{1}{2}\pi,$$

* K is a positive number, not always the same at each occurrence, independent of x and t. See § 2 for a more precise definition.

† If $\mu_1\geqslant\frac{1}{2}\pi$ and $\mu_2\geqslant\frac{1}{2}\pi$, the sector (1·1) includes the half-plane $\mathscr{R}(\kappa t)>K$, and so our hypotheses are not less than before. We do not need to consider this case as it was dealt with fully in Wright (1940). I discuss the two cases in which $\mu_1\geqslant\frac{1}{2}\pi$, $\mu_2<\frac{1}{2}\pi$ or $\mu_1<\frac{1}{2}\pi$, $\mu_2\geqslant\frac{1}{2}\pi$ in § 3, and show that these cannot give anything new.

where $\gamma = \arg \kappa$. Then the sector (1·1) includes all but a finite portion of the positive half of the real axis in the t-plane and lies within the half-plane $\mathscr{R}(\kappa t) > K$. The part of the x-plane in which we find the asymptotic expansion of $f(x)$ is that in which, for a suitable value of $x^{1/\kappa}$,

$$-\min(\mu_1, \omega_2) \leqslant \arg x^{1/\kappa} \leqslant \min(\mu_2, \omega_1), \tag{1·2}$$

where $\omega_1 = \omega(\mu_1, -\gamma)$, $\omega_2 = \omega(\mu_2, \gamma)$ and $\omega(\mu, \gamma)$ is a real transcendental function of μ, γ and $|\kappa|$. The relevant properties of the function ω are exhibited in Lemma 1.

The region (1·2) is bounded by the two spirals (degenerating to straight lines if κ is real) whose equations are

$$\arg x^{1/\kappa} = -\min(\mu_1, \omega_2), \quad \arg x^{1/\kappa} = \min(\mu_2, \omega_1).$$

If $\mu_1 = -\gamma$ or $\mu_2 = \gamma$, then $\omega_1 = \gamma$ or $\omega_2 = -\gamma$, the two spirals coincide and the region (1·2) consists only of the resulting spiral. Hence if we know that $\phi(t)$ is regular and satisfies condition A on the positive half of the real axis, we have the asymptotic expansion of $f(x)$ along the spiral $\arg x^{1/\kappa} = \gamma$.

If $\mathscr{R}(1/\kappa) < \frac{1}{2}$, a number $\mu_0 = \mu_0(\gamma)$ exists such that, if $\mu_2 = \mu_0$ and

$$\mu_1 = 2\pi \mathscr{R}(1/\kappa) - \mu_0,$$

then the region (1·2) consists of the whole plane and we can determine the expansion of $f(x)$ for all large x. We have always $\mu_0 < \frac{1}{2}\pi$ and $2\pi\mathscr{R}(1/\kappa) - \mu_0 < \frac{1}{2}\pi$, so the sector (1·1) still lies within the half-plane $\mathscr{R}(\kappa t) > K$; in fact, the angle of the sector is

$$\mu_1 + \mu_2 = 2\pi \mathscr{R}(1/\kappa) < \pi.$$

Hence when $\mathscr{R}(1/\kappa) < \frac{1}{2}$ we can deduce *all* the results of Wright (1940) from hypotheses substantially less than those of that paper.

Since I now prove part of my former results under wider conditions, it is natural to ask whether the present results are in some sense 'best possible'. This is so and I give examples of functions $\phi(t)$ and $f(x)$ such that $\phi(t)$ is regular and satisfies condition A throughout the sector (1·1), but the asymptotic expansion of $f(x)$ takes a different form just outside the region (1·2).

Except for Lemma 5, this paper can be read entirely independently of Wright (1940).

Notation

2. We write $\delta = |\kappa|$, $\gamma = \arg \kappa \quad (|\gamma| < \frac{1}{2}\pi)$,

and, when $\mathscr{R}(1/\kappa) < \frac{1}{2}$, we define $\mu_0 = \mu_0(\gamma)$ by

$$\tan \mu_0(\gamma) = \frac{e^{2\pi\delta^{-1}\sin\gamma} - \cos(2\pi\delta^{-1}\cos\gamma)}{\sin(2\pi\delta^{-1}\cos\gamma)}, \quad |\mu_0(\gamma)| < \tfrac{1}{2}\pi. \tag{2·1}$$

We write $$\zeta_1 = \zeta_2 = \tfrac{1}{2}\pi \quad (\mathscr{R}(1/\kappa) \geqslant \tfrac{1}{2}),$$

$$\zeta_1 = \mu_0(-\gamma), \quad \zeta_2 = \mu_0(\gamma) \quad (\mathscr{R}(1/\kappa) < \tfrac{1}{2}).$$

We suppose that $\gamma \leqslant \mu < \frac{1}{2}\pi$, write

$$\chi(v, \mu, \gamma) = \cos v - \cos \mu e^{(\mu+v)\tan\mu - 2\pi\delta^{-1}\sec\mu\sin(\mu-\gamma)},$$

and define $\omega = \omega(\mu, \gamma)$ by

$$\chi(\omega, \mu, \gamma) = 0, \quad -\mu \leqslant \omega < \tfrac{1}{2}\pi, \tag{2·2}$$

so that $$\cos\omega = \cos\mu e^{(\mu+\omega)\tan\mu - 2\pi\delta^{-1}\sec\mu\sin(\mu-\gamma)}. \tag{2·3}$$

We prove in Lemma 1 that there is one and only one value of ω satisfying (2·2).

The real numbers μ_1 and μ_2 satisfy

$$-\tfrac{1}{2}\pi < -\mu_1 \leqslant \gamma \leqslant \mu_2 < \tfrac{1}{2}\pi. \tag{2·4}$$

We describe the region (1·1) in the t-plane as the (μ_1, μ_2) sector.

We choose arg x to satisfy

$$-\pi < \arg x - \tan\gamma \log|x| \leqslant \pi,$$

and write $$X = X_0 = x^{1/\kappa}, \quad X_s = Xe^{2\pi si/\kappa},$$

where s is any integer. We use X_S and Y to denote particular X_s. Thus always

$$X^\kappa = X_s^\kappa = X_S^\kappa = Y^\kappa = x,$$

and $$-\pi\delta^{-1}\cos\gamma < \arg X \leqslant \pi\delta^{-1}\cos\gamma. \tag{2·5}$$

We use ϵ and ϵ' to denote positive numbers, to be thought of as small; K is a positive number, not always the same at each occurrence, independent of x (and so of X, X_s and Y), of the real variables r, v and w and of the complex variables u and t, but possibly depending on some or all of

$$\epsilon, \epsilon', \kappa, \beta, \tau, \mu_1, \mu_2, \alpha, \alpha_1, \ldots, \alpha_{M+1}, A_1, \ldots, A_M.$$

We use the notation $\psi = O(\chi)$ to denote that $|\psi| < K|\chi|$ and we always suppose that $|x| > K$.

We say that $\phi(t)$ satisfies 'condition A' in a region in the t-plane if there are an integer $M \geqslant 0$ and numbers $\alpha_1, \ldots, \alpha_{M+1}, A_1, \ldots, A_M$ such that

$$\mathscr{R}(\alpha_1) \leqslant \mathscr{R}(\alpha_2) \leqslant \ldots \leqslant \mathscr{R}(\alpha_M) < \mathscr{R}(\alpha_{M+1}),$$

and $$\frac{\phi(t)}{\Gamma(\kappa t+\beta)} = \sum_{m=0}^{M} \frac{\kappa A_m}{\Gamma(\kappa t+\alpha_m)} + O\left(\frac{1}{\Gamma(\kappa t+\alpha_{M+1})}\right) \tag{2·6}$$

in this region.

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The asymptotic expansion $I(Y)$ is defined by*

$$I(Y) = Ye^Y\left\{\sum_{m=1}^{M} A_m Y^{-\alpha_m} + O(Y^{-\alpha_{M+1}})\right\}.$$

We also write $$S(\kappa, \tau; x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\kappa n+\tau)}, \quad F(Y) = \sum_{n=0}^{\infty} \frac{\psi(n)\, Y^{\kappa n}}{\Gamma(\kappa n+\alpha)},$$

$$J(w) = \int_{K/w}^{\infty} v^{\frac{1}{2}-\tau} e^{w(v-1-v\log v)}\, dv.$$

Statement of results

3. My main results are as follows:

Theorem 1. *If* (i) $-\zeta_1 < -\mu_1 \leqslant \gamma \leqslant \mu_2 < \zeta_2$,

(ii) $\phi(t)$ *is regular and satisfies condition A throughout the* (μ_1, μ_2) *sector*,

and (iii) *some* X_S *satisfies*

$$-\min(\mu_1, \omega_2) \leqslant \arg X_S \leqslant \min(\mu_2, \omega_1), \tag{3·1}$$

then $$f(x) = I(X_S).$$

Theorem 2. *If* (i) $\mathscr{R}(1/\kappa) < \frac{1}{2}$ *and*

(ii) $\phi(t)$ *is regular and satisfies condition A throughout the* $(\mu_0(-\gamma), \mu_0(\gamma))$ *sector, then*

$$f(x) = \sum_{|\arg X_s| < \frac{1}{2}\pi} I(X_s).$$

If, in addition, there is an X_S *such that*†

$$-\mu_0(-\gamma) + K < \arg X_S < \mu_0(\gamma) - K,$$

then $$f(x) = I(X_S).$$

The first part of Theorem 2 is Theorem 2 of Wright (1940), except that in that theorem the function $\phi(t)$ had to satisfy stricter conditions, viz. $\phi(t)$ had to be regular and satisfy condition A throughout the $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ sector, i.e. a half-plane.

The significance of the various conditions of Theorem 1 will be better appreciated if we assume for the moment the following lemma.

* This definition of $I(y)$ is not quite the same as that of Wright (1940), but the difference is immaterial within the range of validity of our results. All our statements remain true if $I(y)$ is given its former, slightly more complicated form.

† This is so for all x except those in the neighbourhood of the spiral

$$\arg x = \tan\gamma \log|x| + \mu_0(\gamma)\,\delta \sec\gamma.$$

Lemma 1. (i) *If* $\gamma \leqslant \mu < \frac{1}{2}\pi$, *there is just one number* ω *satisfying* (2·2). *In particular, if* $\mu = \gamma$, *then* $\omega = -\gamma$; *otherwise* $\omega > -\mu$.

(ii) *If* $\mathscr{R}(1/\kappa) \geqslant \frac{1}{2}$, ω *increases with* μ.

(iii) *If* $\mathscr{R}(1/\kappa) < \frac{1}{2}$, ω *increases with* μ *when* $\gamma < \mu < \mu_0(\gamma)$, *attains its maximum value* $\mu_0(-\gamma)$ *when* $\mu = \mu_0(\gamma)$ *and decreases as* μ *increases when* $\mu_0(\gamma) < \mu < \frac{1}{2}\pi$.

We can now see that we lose nothing by the restriction that $\mu_1 < \zeta_1$ and $\mu_2 < \zeta_2$ in Theorem 1, nor, a fortiori, by the restriction (2·4). Let us suppose first that $\mathscr{R}(1/\kappa) < \frac{1}{2}$, so that $\zeta_1 = \mu_0(-\gamma)$ and $\zeta_2 = \mu_0(\gamma)$. In this case, if $\mu_1 \geqslant \zeta_1$ and $\mu_2 \geqslant \zeta_2$, we take $\mu_1 = \zeta$ and $\mu_2 = \zeta_2$ and have complete information from Theorem 2. On the other hand, if, for example, $\mu_1 \geqslant \zeta_1 = \mu_0(-\gamma)$ and $\mu_2 < \zeta_2 = \mu_0(\gamma)$, we have, by Lemma 1,

$$\omega_2 < \mu_0(-\gamma) \leqslant \mu_1$$

and so

$$\min(\mu_1, \omega_2) = \omega_2.$$

There is therefore no loss in (3·1) if we replace μ_1 by $\omega_2 < \mu_0(-\gamma)$. Similar remarks apply to the case when $\mu_1 < \mu_0(-\gamma)$ and $\mu_2 \geqslant \mu_0(\gamma)$. It is thus clear that we can gain no more information by taking $\mu_1 \geqslant \zeta_1$ or $\mu_2 \geqslant \zeta_2$.

When $\mathscr{R}(1/\kappa) \geqslant \frac{1}{2}$, $\zeta_1 = \zeta_2 = \frac{1}{2}\pi$. If $\mu_1 \geqslant \frac{1}{2}\pi$ and $\mu_2 \geqslant \frac{1}{2}\pi$, the results of Wright (1940) apply. Arguments similar to those used above show that we need not consider the cases $\mu_1 \geqslant \frac{1}{2}\pi$, $\mu_2 < \frac{1}{2}\pi$ and $\mu_1 < \frac{1}{2}\pi$, $\mu_2 \geqslant \frac{1}{2}\pi$. Hence here also the restriction is unimportant.

Finally, we show that our results are 'best possible' and that the conditions that we have laid down cannot be relaxed.

Theorem 3. *Theorems* 1 *and* 2 *are, in a certain sense, best possible results; i.e. the possible interval* (3·1), *of* arg X_S *cannot be enlarged by* ϵ *at one end or the other, nor can the* $(\mu_0(-\gamma), \mu_0(\gamma))$ *sector in Theorem* 2 *be replaced by either of the* $(\mu_0(-\gamma)-\epsilon, \mu_0(\gamma))$ *or* $(\mu_0(-\gamma), \mu_0(\gamma)-\epsilon)$ *sectors. This is true for every* κ, M, μ_1, μ_2 *and positive* ϵ.

I prove this (in § 7) by constructing in each case a suitable $f(x)$ for which the particular result in question is false under the enlarged conditions.*

Proof of preliminary lemmas

4. We now prove six preliminary lemmas, including Lemma 1, which we have already formulated.

Let us suppose that $\gamma \leqslant \mu < \frac{1}{2}\pi$ and that $-\frac{1}{2}\pi \leqslant v \leqslant \frac{1}{2}\pi$. Then $\chi(v, \mu, \gamma)$ and $\partial\chi/\partial v$ are continuous functions of v and

$$\frac{\partial^2 \chi}{\partial v^2} = -\cos v - \sin^2 \mu \sec \mu \, e^{(\mu+v)\tan\mu - 2\pi\delta^{-1}\sec\mu\sin(\mu-\gamma)} \leqslant 0.$$

* In certain cases my $f(x)$ depends on ϵ. I have also found an $f(x)$ independent of ϵ for which the theorems are false under the enlarged conditions for every ϵ. But the proof of this demands a knowledge of the asymptotic expansion of a new type of integral function; as I have not yet published the necessary theorems I content myself with finding a Gegenbeispiel for any given ϵ.

Also
$$\chi(\pm\tfrac{1}{2}\pi, \mu, \gamma) < 0, \quad \chi(-\mu, \mu, \gamma) \gtrless 0,$$
the latter according as $\mu \gtrless \gamma$. Hence, if $\mu > \gamma$, the equation
$$\chi(v, \mu, \gamma) = 0 \tag{4·1}$$
has two roots $v = \omega'$ and $v = \omega$ such that
$$-\tfrac{1}{2}\pi < \omega' < -\mu < \omega < \tfrac{1}{2}\pi. \tag{4·2}$$
On the other hand, if $\mu = \gamma$,
$$\frac{\partial\chi}{\partial v} = -\sin v - \sin\gamma\, e^{(\gamma+v)\tan\gamma},$$
which vanishes for $v = -\gamma$. Hence the equation has just one double root at
$$v = \omega = -\gamma = -\mu. \tag{4·3}$$
We have thus proved (i) of Lemma 1. Also
$$\chi(v, \mu, \gamma) \geqslant 0 \tag{4·4}$$
when $-\mu \leqslant v \leqslant \omega$; in fact, when $-\omega' \leqslant v \leqslant \omega$.

Differentiating (2·3) logarithmically with respect to μ, we obtain
$$\frac{\partial\omega}{\partial\mu} = \frac{\cos\omega}{\cos\mu\sin(\mu+\omega)}\left(\frac{2\pi\cos\gamma}{\delta} - \mu - \omega\right). \tag{4·5}$$
By (4·2) and (4·3), $0 \leqslant \mu+\omega < \pi$. Hence $\partial\omega/\partial\mu > 0$, so long as
$$\mu+\omega < 2\pi\delta^{-1}\cos\gamma,$$
If $\mathscr{R}(1/\kappa) \geqslant \tfrac{1}{2}$, i.e. if $2\pi\delta^{-1}\cos\gamma \geqslant \pi$, we see that ω increases with μ throughout the whole interval $\gamma < \mu < \tfrac{1}{2}\pi$. This is (ii) of Lemma 1.

If $\mathscr{R}(1/\kappa) < \tfrac{1}{2}$, i.e. if $2\pi\delta^{-1}\cos\gamma < \pi$, the position is different. If possible, suppose that
$$\mu+\omega < 2\pi\delta^{-1}\cos\gamma$$
for all μ such that $\gamma < \mu < \tfrac{1}{2}\pi$. Taking logarithms in (2·3) we have
$$\log\cos\omega = \log\cos\mu - \tan\mu\left(\frac{2\pi\cos\gamma}{\delta} - \mu - \omega\right) + \frac{2\pi\sin\gamma}{\delta} \to -\infty$$
as $\mu \to \tfrac{1}{2}\pi$. Since $\partial\omega/\partial\mu > 0$, it follows that $\omega \to \tfrac{1}{2}\pi$ as $\mu \to \tfrac{1}{2}\pi$ and so
$$\mu+\omega \to \pi > 2\pi\delta^{-1}\cos\gamma,$$
a contradiction. Hence for some value of μ $(\gamma < \mu < \tfrac{1}{2}\pi)$ we have $\mu+\omega = 2\pi\delta^{-1}\cos\gamma$ and $\partial\omega/\partial\mu = 0$. By (2·3) this occurs when
$$\mu+\omega = 2\pi\delta^{-1}\cos\gamma, \quad \cos\omega = e^{2\pi\delta^{-1}\sin\gamma}\cos\mu, \tag{4·6}$$
and solving for μ and ω we find that
$$\tan\mu = \tan\mu_0(\gamma), \quad \tan\omega = \tan\mu_0(-\gamma),$$

where $\mu_0(\gamma)$ is defined by (2·1). Since $|\mu|<\frac{1}{2}\pi$ and $|\omega|<\frac{1}{2}\pi$, it follows that

$$\mu=\mu_0(\gamma), \quad \omega=\mu_0(-\gamma). \tag{4·7}$$

Hence when $\mu=\mu_0(\gamma)$ we have the maximum value $\mu_0(-\gamma)$ of ω, and (iii) of Lemma 1 is then obvious by (4·5).

Lemma 2. *If* $\mu_1=-\gamma$ *or* $\mu_2=\gamma$ *(or both), then*

$$-\min(\mu_1,\omega_2)=\gamma=\min(\mu_2,\omega_1).$$

If $-\zeta_1<-\mu_1<\gamma<\mu_2<\zeta_2$, *then*

$$-\min(\mu_1,\omega_2)<\gamma<\min(\mu_2,\omega_1).$$

Lemma 3. *If* $\mathscr{R}(1/\kappa)<\frac{1}{2}$ *then*

$$-\mu_0(-\gamma)=\mu_0(\gamma)-2\pi\delta^{-1}\cos\gamma.$$

Lemma 2 (which gives us a little information about the interval (3·1) and the region (1·2)) is an immediate corollary of Lemma 1. Lemma 3 follows at once from (4·6) and (4·7).

Lemma 4. *If*

$$-\zeta_1+K<\arg X_S<\zeta_2-K, \tag{4·8}$$

then

$$\text{(i)}\quad \mathscr{R}(X_S)>K\,|X_S|, \tag{4·9}$$

$$\text{(ii)}\quad \mathscr{R}(X_s)<\mathscr{R}(X_S)-K\,|X_S| \tag{4·10}$$

when $s\neq S$ *and* $|\arg X_s|\leqslant\frac{3}{2}\pi$, *and*

$$\text{(iii)}\quad \sum_{|\arg X_s|<\frac{1}{2}\pi} I(X_s)=I(X_S). \tag{4·11}$$

(4·9) is immediate, since $|\arg X_S|<\frac{1}{2}\pi-K$, and so

$$\mathscr{R}(X_S)>|X_S|\cos(\tfrac{1}{2}\pi-K)>K\,|X_S|.$$

Again, if $s\neq 0$,

$$|\arg X_s|\geqslant 2\pi\,|s|\,\delta^{-1}\cos\gamma-|\arg X|\geqslant\pi\delta^{-1}\cos\gamma=\pi\mathscr{R}(1/\kappa)$$

by (2·5). Hence, if $\mathscr{R}(1/\kappa)\geqslant\frac{1}{2}$, then $|\arg X_s|\geqslant\frac{1}{2}\pi$ and so (4·8) is only satisfied for $S=0$, since $\zeta_1=\zeta_2=\frac{1}{2}\pi$. The left-hand side of (4·10) must then be negative or zero and (4·10) follows from (4·9). Finally, (4·11) is an identity since the sum on the left-hand side contains just one term, viz. $I(X_S)$.

If $\mathscr{R}(1/\kappa)<\frac{1}{2}$, then $\zeta_1=\mu_0(-\gamma)$ and $\zeta_2=\mu_0(\gamma)$. If $\xi_s=\arg X_s$, it follows from the definitions of X_s and $\mu_0(\gamma)$ that

$$\mathscr{R}(X_s)-\mathscr{R}(X_{s-1})=H_1\,|X_s|\sin(\zeta_2-\xi_s), \tag{4·12}$$

and

$$\mathscr{R}(X_s)-\mathscr{R}(X_{s+1})=H_2\,|X_s|\sin(\xi_s+\zeta_1), \tag{4·13}$$

where $$H_1 = e^{-2\pi\delta^{-1}\sin\gamma}\sin(2\pi\delta^{-1}\cos\gamma)\sec\zeta_2 = K,$$

and $$H_2 = e^{4\pi\delta^{-1}\sin\gamma}H_1 = K.$$

Now let $s_1 < S$ and $\xi_{s_1} > -\frac{1}{2}\pi$. Then $\sin(\zeta_2 - \xi_s) > K$ when $s = s_1+1, \ldots, S$ and so, by continued application of (4·12),

$$\mathscr{R}(X_{s_1}) < \mathscr{R}(X_{s_1+1}) - K\,|X_{s_1+1}| < \mathscr{R}(X_{s_1+2}) - K\,|X_{s_1+2}|$$
$$< \ldots < \mathscr{R}(X_S) - K\,|X_S|,$$

which is (4·10) for $s = s_1$. A similar use of (4·13) proves (4·10) when $s > S$ and $\xi_s < \frac{1}{2}\pi$. Finally, (4·10) is an immediate consequence of (4·9) when $\frac{1}{2}\pi \leqslant |\xi_s| \leqslant \frac{3}{2}\pi$, since $\mathscr{R}(X_s) \leqslant 0$.

From (4·10) we see that all the $I(X_s)$ except $I(X_S)$ on the left hand side of (4·11) can be absorbed in the error term of $I(X_S)$. The truth of (4·11) follows.

Lemma 5. $S(\kappa, \tau; x) = \dfrac{1}{\kappa}\sum_{|\arg X_s| < \frac{1}{2}\pi} X_s^{1-\tau} e^{X_s} + O(1).$

Lemma 6. *If* $$-\zeta_1 + K < \arg Z < \zeta_2 - K \qquad (4·14)$$

then $$\log S(\kappa, \tau; Z^\kappa) \sim Z.$$

Lemma 5 is a trivial corollary of Lemma 6 of Wright (1940), and Lemma 6 follows at once from Lemmas 4 and 5 of the present paper (with Z for X_S).

Lemma 7. *If* $w > K$, *then* $J(w) = O(w^{-\frac{1}{2}})$.

This is part of Lemma 10 of Wright (1940).

The fundamental lemma

5. *Lemma* 8. *If*

$$\text{(i)}\quad -\tfrac{1}{2}\pi < -\mu_1 \leqslant \gamma \leqslant \mu_2 < \tfrac{1}{2}\pi,$$

$$\text{(ii)}\quad -\min(\mu_1, \omega_2) \leqslant \arg Y \leqslant \min(\mu_2, \omega_1), \qquad (5·1)$$

and (iii) $\psi(t)$ *is a regular bounded function of t throughout the* (μ_1, μ_2) *sector, then*

$$F(Y) = O(Y^{1-\alpha}e^Y).$$

We have four possibilities, viz.

$$\text{(i)}\quad -\mu_1 = \gamma = \mu_2, \qquad \text{(ii)}\quad -\mu_1 < \gamma < \mu_2,$$
$$\text{(iii)}\quad -\mu_1 < \gamma = \mu_2, \qquad \text{(iv)}\quad -\mu_1 = \gamma < \mu_2.$$

In each of the cases (i), (iii) and (iv), the condition (5·1) becomes $\arg Y = \gamma$ by Lemma 2; also the (μ_1, μ_2) sector in case (i) forms part of the corresponding sectors in cases (iii) and (iv). Hence Lemma 8 in cases (iii) and (iv) is a corollary of the same lemma in case (i) and we have only to prove the lemma in cases (i) and (ii).

Proof of Lemma 8 *when* $-\mu_1 = \gamma = \mu_2$. In this case $\arg Y = \gamma$ and $|\psi(t)| < K$ when t is real and $t > K$. When $|\arg(\kappa t)| < \pi$ and $|\arg(\kappa t+\alpha)| < \pi$, we have

$$\left|\frac{1}{\Gamma(\kappa t+\alpha)}\right| < K\left|t^{\frac{1}{2}-\alpha}\left(\frac{e}{\kappa t}\right)^{\kappa t}\right|. \tag{5·2}$$

Hence, when $n > K$,
$$\left|\frac{\psi(n)\, Y^{\kappa n}}{\Gamma(\kappa n+\alpha)}\right| < K\vartheta(n),$$

where
$$\vartheta(t) = t^{\frac{1}{2}-\mathscr{R}(\alpha)}(e\,|\,Y\,|\,\delta^{-1}\,t^{-1})^{\delta t\cos\gamma},$$

and so
$$F(Y) < K\,|\,Y\,|^{K} + K\sum_{n>\kappa}\vartheta(n). \tag{5·3}$$

Differentiating $\vartheta(t)$ logarithmically we have

$$\frac{1}{\vartheta(t)}\frac{d}{dt}\vartheta(t) = \frac{\frac{1}{2}-\mathscr{R}(\alpha)}{t} + \delta\cos\gamma\,\log\left(\frac{|\,Y\,|}{\delta t}\right).$$

The latter expression is positive when $t < t_0$, vanishes when $t = t_0$, and is negative when $t > t_0$, where
$$t_0 = |\,Y\,|\,\delta^{-1}\{1+O(Y^{-1})\}.$$

Hence
$$\sum_{n>K}\vartheta(n) \leqslant \vartheta(t_0) + \int_K^\infty \vartheta(t)\,dt$$
$$= O(Y^{\frac{1}{2}-\alpha}\,e^{Y}) + \int_K^\infty \vartheta(t)\,dt. \tag{5·4}$$

Putting $t = |\,Y\,|\,\delta^{-1}v$ in this integral, we have

$$\int_K^\infty \vartheta(t)\,dt = K\,|\,Y^{\frac{3}{2}-\alpha}\,e^{Y}\,|\,J\{\mathscr{R}(Y)\}$$
$$= O(Y^{1-\alpha}\,e^{Y}) \tag{5·5}$$

by Lemma 7, since $\mathscr{R}(Y) = |\,Y\,|\cos\gamma > K\,|\,Y\,| > K$. Hence
$$F(Y) = O(Y^{1-\alpha}\,e^{Y}) + O(Y^{K}) = O(Y^{1-\alpha}\,e^{Y})$$

by (5·3), (5·4) and (5·5).

Proof of Lemma 8 *when* $-\mu_1 < \gamma < \mu_2$. We write $\eta = \arg Y$; then
$$\cos\eta \geqslant \cos\{\max(\mu_1, \mu_2)\} > K$$

by conditions (i) and (ii) of Lemma 8. We take a new complex variable u and write

$$r = |\,u\,|,\quad \theta = \arg u,\quad g(u) = \frac{\psi(u/\kappa)\,Y^{u}}{\Gamma(u+\alpha)},$$

$$g_1(u) = \frac{g(u)}{e^{2\pi i u/\kappa}-1},\quad g_2(u) = \frac{g(u)}{1-e^{-2\pi i u/\kappa}},$$

$$v = v(\theta, r, Y) = \cos\theta\,\log(e\,|\,Y\,|/r) - (\eta-\theta)\sin\theta.$$

We write also $h = \delta \cos \gamma (N_1 - \frac{1}{2})$, where N_1 is a fixed positive integer chosen large enough to ensure that $h > 1 - \mathscr{R}(\alpha)$ and that $\psi(u/\kappa)$ is regular and bounded when

$$r \cos \theta \geqslant h, \quad -\mu_1 \leqslant \theta \leqslant \mu_2. \tag{5·6}$$

Then h is a fixed number of the type of K. Also $g(u)$ is regular and, by (5·2),

$$|\, g(u) \,| < K r^{\frac{1}{2} - \mathscr{R}(\alpha)} e^{\nu r} \tag{5·7}$$

throughout the region (5·6).

We take a large positive integer N (which will subsequently be supposed to tend to infinity) and define the following contours in the u-plane:

(i) $\mathscr{A}$ and $\mathscr{A}'$ are the segments of the straight line $\mathscr{R}(u) = h$ on which $-\mu_1 \leqslant \theta \leqslant \mu_2$ and $-\mu_1 \leqslant \theta \leqslant \eta$ respectively;

(ii) $\mathscr{B}_N$ and $\mathscr{B}_N'$ are the arcs of the circle $|\, u \,| = \delta(N + \frac{1}{2})$ on which $-\mu_1 \leqslant \theta \leqslant \mu_2$ and $-\mu_1 \leqslant \theta \leqslant \eta$ respectively;

(iii) $\mathscr{C}_N$, $\mathscr{D}_N$, $\mathscr{E}_N$ are the segments of the lines $\theta = -\mu_1$, $\theta = \mu_2$, $\theta = \eta$ which lie between $\mathscr{A}$ and $\mathscr{B}_N$;

(iv) $\mathscr{C}$, $\mathscr{D}$, $\mathscr{E}$ are the semi-infinite parts of the same lines which lie to the right of $\mathscr{A}$. We note that $\mathscr{C}_N$, $\mathscr{D}_N$, $\mathscr{E}_N$ become $\mathscr{C}$, $\mathscr{D}$, $\mathscr{E}$ when $N \to \infty$. The positive directions of $\mathscr{A}$, $\mathscr{A}'$, $\mathscr{B}_N$, $\mathscr{B}_N'$ are upwards and the positive directions of $\mathscr{C}_N$, $\mathscr{D}_N$, $\mathscr{E}_N$, $\mathscr{C}$, $\mathscr{D}$, $\mathscr{E}$ are outwards from the origin.

Obviously (5·6) is satisfied on all these contours and throughout the two regions enclosed by $\mathscr{A}$, $\mathscr{B}_N$, $\mathscr{C}_N$, $\mathscr{D}_N$ and by $\mathscr{A}'$, $\mathscr{B}_N'$, $\mathscr{C}_N$, $\mathscr{E}_N$ respectively. Hence $g(u)$ is regular throughout these regions and on their boundaries. Also $g_2(u)$ is regular throughout the first region and on its boundary except for simple poles at the zeros of $1 - e^{-2\pi i u/\kappa}$, i.e. at the points $u = \kappa n$ $(N_1 \leqslant n \leqslant N)$, where n is integral; since $-\mu_1 < \gamma < \mu_2$ and N_1 and N are integers, none of these poles lie on the boundary of the first region.

Hence by Cauchy's theorem

$$\left(-\int_{\mathscr{A}} + \int_{\mathscr{B}_N} + \int_{\mathscr{C}_N} - \int_{\mathscr{D}_N} \right) g_2(u)\, du = \kappa \sum_{n=N_1}^{N} g(\kappa n), \tag{5·8}$$

$$\left(-\int_{\mathscr{A}'} + \int_{\mathscr{B}_N'} + \int_{\mathscr{C}_N} - \int_{\mathscr{E}_N} \right) g(u)\, du = 0. \tag{5·9}$$

Now on $\mathscr{A}$ and $\mathscr{A}'$

$$g(u) = O(Y^h), \quad |\, 1 - e^{-2\pi i u/\kappa} \,| > K, \quad g_2(u) = O(Y^h);$$

hence

$$\int_{\mathscr{A}} g_2(u)\, du = O(Y^h), \quad \int_{\mathscr{A}'} g(u)\, du = O(Y^h).$$

When $\cos \theta > K$ and r is sufficiently large, i.e. if $r > K \,|\, Y \,|$, we have $\nu < -K \log r$. Also on $\mathscr{B}_N$ and $\mathscr{B}_N'$

$$|\, 1 - e^{-2\pi i u/\kappa} \,| > K$$

and so, if N is sufficiently large,

$$|g(u)| < Ke^{-KN\log N}, \quad |g_2(u)| < Ke^{KN\log N}.$$

Hence
$$\int_{\mathscr{B}_N} g_2(u)\,du \to 0, \quad \int_{\mathscr{B}'_N} g(u)\,du \to 0,$$
as $N \to \infty$.

Subtracting (5·9) from (5·8), letting $N \to \infty$ and noting that $g_1(u) = g_2(u) - g(u)$, we have

$$\kappa F(Y) = \int_{\mathscr{C}} g_1(u)\,du - \int_{\mathscr{D}} g_2(u)\,du + \int_{\mathscr{E}} g(u)\,du + O(Y^h)$$
$$= L_1 - L_2 + L + O(Y^h) \quad \text{(say)}. \tag{5·10}$$

On $\mathscr{E}$ we have
$$\theta = \eta, \quad v = \cos\eta \log(e|Y|/r),$$
$$|g(u)| < Kr^{\frac{1}{2}-\mathscr{R}(\alpha)}\, e^{r\cos\eta\log(e|Y|/r)}$$
by (5·7). Using this upper bound in L and putting $r = |Y|v$, we have

$$|L| < K|Y|^{\frac{3}{2}-\mathscr{R}(\alpha)}\, e^{|Y|\cos\eta} J(|Y|\cos\eta) = O(Y^{1-\mathscr{R}(\alpha)}\, e^Y), \tag{5·11}$$

by Lemma 7, since
$$\mathscr{R}(Y) = |Y|\cos\eta > K|Y| > K. \tag{5·12}$$

On $\mathscr{C}$, $\theta = -\mu_1$ and
$$|e^{2\pi iu/\kappa} - 1| > Ke^{2\pi\delta^{-1} r\sin(\mu_1+\gamma)},$$
since $\gamma > -\mu_1$; hence
$$|g_1(u)| < Kr^{\frac{1}{2}-\mathscr{R}(\alpha)}\, e^{\nu_1 r},$$
where
$$\nu_1 = \nu(-\mu_1) - 2\pi\delta^{-1}\sin(\mu_1+\gamma)$$
$$= \cos\mu_1 \log\{e|Y|\lambda_1(\eta)/r\},$$
and
$$\lambda_1(\eta) = \exp\{(\mu_1+\eta)\tan\mu_1 - 2\pi\delta^{-1}\sec\mu_1\sin(\mu_1+\gamma)\} > K.$$

If we put $r = |Y|\lambda_1(\eta)v$ in L_1, we have

$$\nu_1 r = |Y|\lambda_1(\eta)\cos\mu_1 v(1-\log v),$$

and
$$L_1 = O(Y^{\frac{3}{2}-\alpha}\, e^{|Y|\lambda_1(\eta)\cos\mu_1} J\{|Y|\lambda_1(\eta)\cos\mu_1\})$$
$$= O(Y^{1-\alpha}\, e^{|Y|\lambda_1(\eta)\cos\mu_1}),$$
by Lemma 7. Now
$$\lambda_1(\eta)\cos\mu_1 = \cos\eta - \chi(\eta, \mu_1, -\gamma) \leqslant \cos\eta,$$
by (4·4), provided that $-\mu_1 \leqslant \eta \leqslant \mu_2$, which is certainly true when (5·1) is satisfied. Hence
$$L_1 = O(Y^{1-\alpha}\, e^Y). \tag{5·13}$$

If we use
$$\lambda_2(\eta) = \exp\{(\mu_2-\eta)\tan\mu_2 - 2\pi\delta^{-1}\sec\mu_2\sin(\mu_2-\gamma)\}$$
similarly in L_2, we have

$$\lambda_2(\eta)\cos\mu_2 = \cos\eta - \chi(-\eta, \mu_2, \gamma) \leqslant \cos\eta$$

when $-\omega_2 \leqslant \eta \leqslant \mu_2$ and
$$L_2 = O(Y^{1-\alpha}\, e^Y). \tag{5·14}$$

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By (5·10)–(5·14) we have

$$F(Y) = O(Y^{1-\alpha}e^{Y}) + O(Y^{h}) = O(Y^{1-\alpha}e^{Y}).$$

Proof of Theorems 1 and 2

6. If we put $\alpha = \alpha_{M+1}$ and

$$\psi(t) = \Gamma(\kappa t + \alpha)\left\{\frac{\phi(t)}{\Gamma(\kappa t+\beta)} - \sum_{m=1}^{M}\frac{\kappa A_m}{\Gamma(\kappa t+\alpha_m)}\right\},$$

we have by (2·6)

$$f(x) = \kappa \sum_{m=1}^{M} A_m S(\kappa, \alpha_m; x) + F(Y)$$

$$= \sum_{|\arg X_s|<\frac{1}{2}\pi} I(X_s) + F(Y) + O(1),$$

where Y is any κth root of x.

To prove Theorem 1 we have only to put $Y = X_S$ in Lemma 8. The conditions of that lemma are satisfied when those of Theorem 1 are satisfied, and

$$f(x) = \sum_{|\arg X|<\frac{1}{2}\pi} I(X_s) + O(X_S{}^{1-\alpha}e^{X_S}) + O(1)$$

$$= I(X_S)$$

by Lemma 4 (i) and (iii).

To prove Theorem 2 we put Y for that X_s which satisfies

$$-\mu_0(-\gamma) < \arg X_s \leqslant \mu_0(\gamma).$$

There is just one such X_s, since

$$\arg X_{s+1} - \arg X_s = 2\pi\delta^{-1}\cos\gamma = \mu_0(-\gamma) + \mu_0(\gamma)$$

by Lemma 3. We also put $\mu_1 = \mu_0(-\gamma)$, $\mu_2 = \mu_0(\gamma)$ in Lemma 8, so that

$$\omega_1 = \omega(\mu_0(-\gamma), -\gamma) = \mu_0(\gamma), \quad \omega_2 = \mu_0(-\gamma)$$

by Lemma 1. Again the conditions of Lemma 8 are satisfied and so

$$f(x) = \sum_{|\arg X|<\frac{1}{2}\pi} I(X_s) + O(Y^{1-\alpha}e^{Y}) + O(1).$$

Now $\quad |\arg Y| < \max\{\mu_0(\gamma), \mu_0(-\gamma)\} < \frac{1}{2}\pi - K, \quad \mathscr{R}(Y) > K|Y|,$

and so $O(1) = O(Y^{1-\alpha}e^{Y})$ and Y is one of the X_s for which $|\arg X_s| < \frac{1}{2}\pi$. Hence $O(1)$ and $O(Y^{1-\alpha}e^{Y})$ can be absorbed in the error term of $I(Y)$ and

$$f(x) = \sum_{|\arg X_s|<\frac{1}{2}\pi} I(X_s).$$

This is the first part of Theorem 2 and the second part follows by Lemma 4.

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PROOF OF THEOREM 3

7. The proof of Theorem 3 is based on three lemmas.

Lemma 9. *If* $0<C<\pi$, $-\tfrac{1}{2}\pi<\mu<\tfrac{1}{2}\pi$, *and* λ *is defined by*

$$\tan\lambda = \frac{e^{C\tan\mu}-\cos C}{\sin C}, \quad |\lambda|<\tfrac{1}{2}\pi,$$

then

$$\mu<\lambda<\mu+C.$$

Let

$$\varpi(v) = \cos\mu-\cos(\mu+v)\,e^{v\tan\mu}.$$

Then

$$\frac{\partial\varpi}{\partial v} = \sec\mu\sin v\,e^{v\tan\mu},$$

and so $\varpi(v)$ has a minimum when $v=0$ and

$$\varpi(v)>\varpi(O) = 0 \quad (-\pi<v<\pi, \quad v\neq 0).$$

Now

$$\sin(\lambda-\mu) = \cos\lambda(\tan\lambda\cos\mu-\sin\mu)$$

$$= \cos\lambda\operatorname{cosec} Ce^{C\tan\mu}\varpi(-C)>0,$$

and

$$\sin(C+\mu-\lambda) = \cos\lambda\{\sin(C+\mu)-\cos(C+\mu)\tan\lambda\}$$

$$= \cos\lambda\operatorname{cosec} C\varpi(C)>0.$$

Since

$$-\pi<C+\mu-\lambda<2\pi, \quad -\pi<\lambda-\mu<\pi,$$

it follows that

$$\lambda-\mu>0, \quad C+\mu-\lambda>0, \quad \mu<\lambda<\mu+C.$$

Lemma 10. *If* $\gamma\leqslant\mu_2<\zeta_2$, *if* $Y^\kappa = x$, *if* ϵ' *is any sufficiently small positive number and if, in particular*, $\mu_2+\epsilon'<\zeta_2$ *and* $\omega_2+\epsilon'<\zeta_1$, *then we can find functions*

$$f_1(x) = \sum_{n=0}^{\infty}\frac{\phi_1(n)\,x^n}{\Gamma(\kappa n+\beta)}, \quad f_2(x) = \sum_{n=0}^{\infty}\frac{\phi_2(n)\,x^n}{\Gamma(\kappa n+\beta)},$$

such that (i) $\phi_1(t)$ *and* $\phi_2(t)$ *are regular and bounded in the* (ζ_1, μ_2) *sector and* (ii)

$$\mathscr{R}\{\log f_1(x)-Y\}>K|Y| \text{ when } \arg Y = \mu_2+\epsilon',$$

$$\mathscr{R}\{\log f_2(x)-Y\}>K|Y| \text{ when } \arg Y = -\omega_2-\epsilon'.$$

We take

$$f_1(x) = S(\kappa,\beta;Z_1^\kappa), \quad f_2(x) = S(\kappa,\beta;Z_2^\kappa),$$

where

$$C_1 = \tfrac{1}{2}\epsilon', \quad C_2 = 2\pi\delta^{-1}\cos\gamma-\mu_2-\omega_2>0,$$

$$Z_1 = Ye^{-iC_1(1-i\tan\mu_2)}, \quad Z_2 = Ye^{2\pi i/\kappa-iC_2(1-i\tan\mu_2)}.$$

Since $\mu_2 < \zeta_2$, it follows from the arguments used in the proof of Lemma 1 that $\mu_2 + \omega_2 < 2\pi\delta^{-1} \cos \gamma$ and so $C_2 > 0$. Now

$$\phi_1(n) = Z_1^{\kappa n} x^{-n} = e^{-i\kappa n C_1(1-i\tan\mu_2)},$$

$$\phi_2(n) = Z_2^{\kappa n} x^{-n} = e^{-i\kappa n C_2(1-i\tan\mu_2)},$$

and so, for $s = 1$ or 2,

$$|\phi_s(t)| = |e^{-iC_s\kappa t(1-i\tan\mu_2)}|$$

$$= \exp[|\kappa t| C_s \sec\mu_2 \sin\{\arg(\kappa t) - \mu_2\}] < 1,$$

provided that $-\pi + \mu_2 \leqslant \arg(\kappa t) \leqslant \mu_2$. Hence (i) of Lemma 10 is satisfied.

If we take $\arg Y = \mu_2 + \epsilon'$, we have

$$\arg Z_1 = \arg Y - C_1 = \mu_2 + C_1 = \mu_2 + \tfrac{1}{2}\epsilon',$$

and so $\arg Z_1$ satisfies (4·14). Hence $\log f_1(x) \sim Z_1$ by Lemma 6 and

$$\mathscr{R}\{\log f_1(x) - Y\} \sim \mathscr{R}(Z_1 - Y)$$

$$= |Y|\{e^{-C_1\tan\mu_2}\cos(\mu_2 + \epsilon - C_1) - \cos(\mu_2 + \epsilon)\}$$

$$= |Y| e^{-C_1\tan\mu_2} \sin C_1 \sec\lambda \sin(\mu + 2C_1 - \lambda)$$

$$\geqslant |Y| e^{-C_1\tan\mu_2} \sin^2 C_1 \sec\lambda = K|Y|$$

by Lemma 9 and since $C_1 = \tfrac{1}{2}\epsilon' < \tfrac{1}{2}\pi$.

If we take $\arg Y = -\omega_2 - \epsilon'$, we have

$$\arg Z_2 = 2\pi\delta^{-1}\cos\gamma - \omega_2 - \epsilon' - C_2 = \mu_2 - \epsilon',$$

so that the condition of Lemma 6 is satisfied when $Z = Z_2$; hence $\log f_2(x) \sim Z_2$ by Lemma 6. Also

$$Z_2 = |Y| e^{i(\mu_2 - \epsilon') + (\mu_2 + \omega_2)\tan\mu_2 - 2\pi\delta^{-1}\sec\mu_2 \sin(\mu_2 - \gamma)}$$

$$= \cos\omega_2 \sec\mu_2 |Y| e^{i(\mu_2 - \epsilon')}$$

by (2·3). Hence

$$\mathscr{R}\{\log f_2(x) - Y\} \sim \mathscr{R}(Z_2 - Y)$$

$$= |Y|\{\cos\omega_2 \sec\mu_2 \cos(\mu_2 - \epsilon') - \cos(\omega_2 + \epsilon')\}$$

$$= |Y| \sec\mu_2 \sin\epsilon' \sin(\mu_2 + \omega_2).$$

If $\mu_2 > \gamma$, we have $\mu_2 + \omega_2 > 0$ by Lemma 1 and so

$$\mathscr{R}\{\log f_2(x) - Y\} > K|Y|.$$

If $\mu_2 = \gamma$, then $\omega_2 = -\gamma$ and $\mu_2 + \omega_2 = 0$. We take $\epsilon'' = \tfrac{1}{2}\epsilon'$, write $\omega_2'' = \omega_2 + \epsilon''$ and define μ_2'' by

$$\omega(\mu_2'', \gamma) = \omega_2'', \quad \gamma < \mu_2'' < \zeta_2.$$

By Lemma 1 this defines just one value of μ_2''. Since $\mu_2''>\gamma$, we have already proved the second part of Lemma 10 when μ_2'', ω_2'' and ϵ'' replace μ_2, ω_2 and ϵ'. Hence we can find an $f_2(x)$ such that $\phi_2(t)$ is regular and bounded throughout the (ζ_1, μ_2'') sector and so, a fortiori, throughout the (ζ_1, μ_2) sector, and such that

$$\mathscr{R}\{\log f_2(x)-Y\}>K\,|\,Y\,|$$

when $\arg Y=-\omega_2''-\epsilon''=-\omega_2-2\epsilon''=-\omega_2-\epsilon'$. This completes the proof of Lemma 10.

Lemma 11. *If* $-\gamma\leqslant\mu_1<\zeta_1$, *if* $Y^\kappa=x$, *if* ϵ' *is any sufficiently small positive number and if, in particular,* $\mu_1+\epsilon'<\zeta_1$ *and* $\omega_1+\epsilon'<\zeta_2$, *then we can find functions*

$$f_3(x)=\sum_{n=0}^{\infty}\frac{\phi_3(n)\,x^n}{\Gamma(\kappa n+\beta)},\quad f_4(x)=\sum_{n=0}^{\infty}\frac{\phi_4(n)\,x^n}{\Gamma(\kappa n+\beta)},$$

such that (i) $\phi_3(t)$ *and* $\phi_4(t)$ *are regular and bounded in the* (μ_1, ζ_2) *sector and* (ii)

$$\mathscr{R}\{\log f_3(x)-Y\}>K\,|\,Y\,|\ \textit{when}\ \arg Y=-\mu_1-\epsilon',$$

$$\mathscr{R}\{\log f_4(x)-Y\}>K\,|\,Y\,|\ \textit{when}\ \arg Y=\omega_1+\epsilon'.$$

The proof of this is precisely similar to that of Lemma 10 and so I omit it.

Now let us suppose that $M=0$ in Theorem 1 and that the interval (3·1) of $\arg X_S$ is replaced by

$$-\min(\mu_1, \omega_2)\leqslant\arg X_S\leqslant\min(\mu_2, \omega_1)+\epsilon. \tag{7·1}$$

Let us suppose that $\omega_1\geqslant\mu_2$. Then we have only to choose ϵ' in Lemma 10 so that, in addition to satisfying the requirements of that lemma, we have $\epsilon'<\epsilon$. If we take $f_1(x)$ for $f(x)$ and $\arg Y=\mu_2+\epsilon'$, we have $Y=X_S$ and

$$\mathscr{R}\{\log f(x)-X_S\}>K\,|\,X_S\,|. \tag{7·2}$$

But, if Theorem 1 is true when (3·1) is replaced by (7·1), we must have

$$f(x)=O(X_S^{1-\beta}e^{X_S}).$$

This is false by (7·2).

If $\omega_1<\mu_2$, we use Lemma 11 and $f_4(x)$ to obtain our contradiction. Again, if (3·1) is replaced by

$$-\min(\mu_1, \omega_2)-\epsilon\leqslant\arg X_S\leqslant\min(\mu_2, \omega_1),$$

we use $f_2(x)$ or $f_3(x)$ as the case may be.

Next we suppose that $M=0$ in Theorem 2 and that $\phi(t)$ is to be regular and satisfy condition A throughout the $(\mu_0(-\gamma), \mu_0(\gamma)-\epsilon)$ sector. We put $\mu_2=\mu_0(\gamma)-\epsilon$ and $\epsilon'<\epsilon$

in Lemma 10. Then $\phi_1(t)$ satisfies condition A throughout the $(\mu_0(-\gamma),\ \mu_0(\gamma)-\epsilon)$ sector and

$$\mathscr{R}\{f_1(x)-Y\}>K\,|\,Y\,|$$

when $\arg Y=\mu_0(\gamma)-\epsilon+\epsilon'<\mu_0(\gamma)$. This contradicts the result of Theorem 2. If the sector $(\mu_0(-\gamma),\ \mu_0(\gamma))$ is replaced by $(\mu_0(-\gamma)-\epsilon,\ \mu_0(\gamma)),\ f_3(x)$ supplies a similar Gegenbeispiel.

Hence Theorem 3 is true when $M=0$; its truth for $M>0$ follows by an obvious use of Lemmas 4 and 5.

Reference

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